φ = Latitude
δ= Solar Declination for the Given Day
t = Solar Hour-Angle Past Noon (15°/hr.)

$$L = \sin^{-1}\left(-0.5\sqrt{2}\cos\varphi\right)$$
$$T = \sin^{-1}\left(0.5\sqrt{2}/\cos L\right)$$
$$T = \cot^{-1}\left(\sin\varphi\right)$$
$$A = \tan^{-1}\left(0.5\sqrt{2}/\tan\varphi\right)$$

$$q = \cos L\cos\left(T-t\right) + \sin L\tan\delta$$
$$m = \frac{\sin(T-t)}{q}$$
$$n = \frac{\sin L\cos(T-t) - \cos L\tan\delta}{q}$$

$$x = 1 - m\cos A - n\sin A$$
$$y = m\sin A - n\cos A$$

Distance from noon to reading point = $\sqrt{2}$

Solving the Spherical Triangle

Fred Sawyer

I had mixed feelings when my daughter began studying trigonometry in high school. It certainly is true that the formulas of gnomonics are replete with the basic trig functions, so any modern approach to dialing should have such a course as a prerequisite. However, in reviewing her textbook I found that the course stopped short of dealing with the spherical trigonometry which provides the modern rationale for most of the gnomonic formulas.

A trip to the library established that none of the newer texts on the shelves included anything but a cursory mention of spherical trigonometry. The older texts, closer to the 19th than to the 21st century, provided more depth in their treatment but reworked all the formulas and presented them in formats designed for evaluation by log tables or slide rules (lamentable casualties of the electronic age).

My goal in this article is to present enough basic information on spherical trigonometry to permit the reader (familiar with plane trig) to solve any spherical triangle and to derive many of the essential equations of spherical astronomy and gnomonics.

Consider a sphere, and draw an arc on its surface. If the arc is part of a circle whose center is also the center of the sphere itself, then the arc is part of a great circle. For example, on the earth, any piece of the equator or a line of longitude is a great circle arc, since the center of the circle from which it comes is the center of the earth. However, any non-equatorial latitude line is part of a minor circle which does not share a center with the globe and is therefore not a great circle arc.

A spherical triangle is a triangle on the surface of a sphere, each of whose sides is an arc (less than 180 degrees) of a great circle. Any three non-collinear points on the sphere's surface define a triangle.

Lengths of the sides of the triangle are given in angular measure; each side's length equals the measure (between 0º and 180º) of the angle formed by its end points with the sphere's center as vertex.

Spherical triangles satisfy the following five rules:

1. The sum of the lengths of a spherical triangle's sides is always less than 360º.

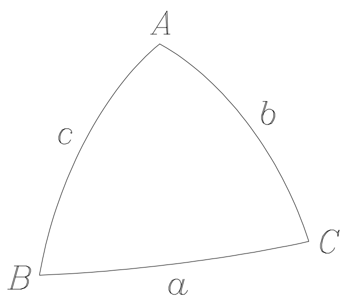
2. The sum of the angles at its vertices is greater than 180º and less than 540º.

3. The sum of the lengths of any two sides is greater than the length of the third side.

4. If a side (or angle) differs from 90º by more than another side (or angle) on a spherical triangle, then it is in the same quadrant as its opposite angle (or side). In other words, they are either both greater than 90º or both less than 90º.

5. Half the sum of two sides of a spherical triangle must be in the same quadrant as half the sum of the two opposite angles.

Every spherical triangle consists of 6 elements: three sides a,b,c and the angles A,B,C opposite them. In general, if any 3 of these elements are given, it is possible to solve for the other 3. To see how this solution is carried out, we must consider 6 cases:



Case I Given 3 sides a,b,c.

In this case, apply the cosine law for sides, the fundamental equation of spherical trigonometry:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad \textit{(1)}$$

This equation allows us to solve for a 4th element A. The remaining two B and C are obtained from analogous forms of the cosine law:

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \quad \textit{(2)}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad \textit{(3)}$$

Case II Given 3 angles A,B,C.

Here we proceed as before, but we use the cosine law for angles:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \quad (4)$$

Note that by specifying the three vertex angles, we uniquely determine a spherical triangle; this is very different from the situation with plane triangles.

Case III Given 2 sides a,b, and their included angle C.

In this case we can apply the cosine law for sides (equation 3 above) to obtain a value for c. We now have 3 sides known and can continue as in Case I.

Case IIII Given 2 angles A,B and their included side c.

By applying equation (4) above we find the third angle C and can proceed as in Case II.

Case V Given 2 sides a,b and the angle A opposite one of them.

First, apply another fundamental rule, the sine law:

$$\sin a / \sin A = \sin b / \sin B = \sin c / \sin C \quad \textit{(5)}$$

From this, we can obtain a value for B. Note, however, that there are 2 possible values of B which will result from this law (one greater than 90º and one less than 90º).

At this point, we apply the rules 4-5 given above, to see if they are sufficient to determine a unique value (i.e. specify the correct quadrant) for B. In the event that these two rules do not uniquely identify B, then the given triangle does actually have 2 solutions.

To finish solving this triangle, obtain c and C by applying the equations known as Napier's analogies:

$$\tan 0.5c = \tan 0.5(a-b) \sin 0.5(A+B) / \sin 0.5(A-B)$$

$$\tan 0.5c = \tan 0.5(a+b) \cos 0.5(A+B) / \cos 0.5(A-B)$$

$$\cot 0.5C = \tan 0.5(A-B) \sin 0.5(a+b) / \sin 0.5(a-b)$$

$$\cot 0.5C = \tan 0.5(A+B) \cos 0.5(a+b) / \cos 0.5(a-b)$$

Case VI Given 2 angles A,B and the side a opposite one of them.

Proceed as in the prior case, applying the sine law (5) and rules 4-5 to obtain either one or two values for side b. Apply Napier's analogies as necessary to complete the solution.

Applications

Suppose now that we consider a specific spherical triangle PQS whose vertices on the celestial sphere are P (the North Pole), Q (the zenith point directly overhead from our location) and S (the location of the sun at this moment in time).
Given this configuration for our triangle we know:

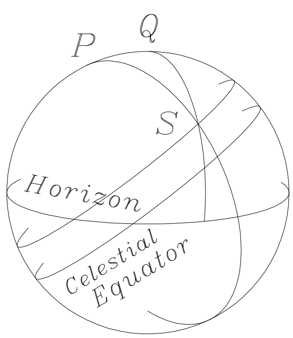
Angle P = the solar hour-angle T *(T)*

Angle Q = 180º - the solar azimuth *(180º-Z)*

Side s = 90º - the Latitude *(90º-L)*

Side q = 90º - the solar declination *(90º-D)*

Side p = 90º - the solar altitude *(90º-A)*



We can now apply the cosine law to obtain

$$\cos p = \cos q \cos s + \sin q \sin s \cos P$$

With substitution (e.g., cos p = cos (90º-A) = sin A), this becomes:

$$\sin A = \sin D \sin L + \cos D \cos L \cos T \qquad (6)$$

Thus, the familiar equation for the sun's altitude, expressed in terms of latitude, declination and hour-angle is simply an example of the cosine law for sides. This is the basic formula underlying virtually all altitude sundials, including capuchin, Regiomontanus and shepherd's dials.

By applying the sine law we can relate altitude & azimuth to declination & time:

$$\sin p / \sin P = \sin q / \sin Q \implies \cos A \sin Z = \cos D \sin T \text{ .}$$

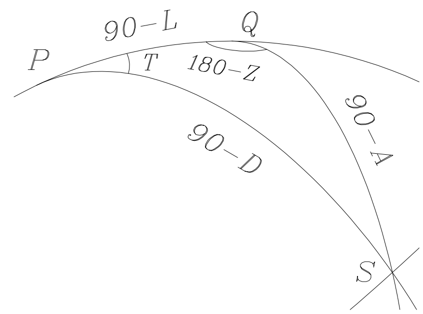
This equation underlies the latitude-independent sundial designed by J.G. Freeman (see *Journal of the Royal Astronomical Society of Canada* for 1978, or *Bulletin of the British Sundial Society* 91(1):18-28.)

By combining equations we can obtain other useful forms. For example, begin with the following formula obtained from an application of the cosine law:

$$\sin D = \sin L \sin A - \cos L \cos A \cos Z$$

Now replace sin A by (sin L sin D + cos L cos D cos T) from equation (6) and, from equation (7), replace cos A by (cos D sin T / sin Z).

Rearranging the resulting equation in order to isolate the variable Z yields:



$$\cot Z = (\sin L \cos T - \cos L \tan D) / \sin T$$

This is the usual formula for the sun's azimuth and for the placing of the qibla line on Islamic dials. It is also the basic equation which is realized graphically in an analemmatic sundial.

In an upcoming article, we will discuss how these basic equations can simplify dealing with the design of reclining/declining dials. In that same article, we will consider how to determine the hours during which the sun shines on any given plane - or, in the parlance of the golden age of dialing: *"a plane howsoever scituated"*.

In the meantime, we conclude with the following list of equations for an arbitrary spherical triangle, all of which are given above or are derivable from this discussion:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$\sin a / \sin A = \sin b / \sin B = \sin c / \sin C$$

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$$

$$\cos a \cos C = \sin a \cot b - \sin C \cot B$$

A: To the Kaaba and its Black Stone in the city of Mecca.