

A Three-Point Sundial Construction

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Projective Dialling, the 9th tract of William Leybourn's book *Dialling* (orig. 1682, 2nd ed. 1700), is attributed by Leybourn to the brilliant seventeenth-century dialist Samuel Foster. The tract contains several constructions, the most interesting of which appears as Chapter 11:

To draw a Dial upon a flat Superficies by means of Three Shadows of a Stile, caused by the Sun upon the same Superficies in one Day, without knowing either the Sun's Declination, the Elevation of the Pole, or Situation of the Plain.

Thus the construction posits that the dialist is given a plane, a stile, and three shadow-points made by the end of the stile on the plane at different times in the course of a single day. From this information one must construct a dial on the plane. To be clear about what information the dialist does not have, it should be pointed out that the construction assumes no knowledge of the geographic latitude of the dial, the reclination or declination of the plane, the date or times at which the shadows were cast, or the placement of the meridian either for the plane or for the geographic location; it is not permitted to take any more readings relating to the location of the sun.

This three-point construction is also discussed by John Collins in his 1659 work *Geometrical Dyalling*. Collins attributes this construction to M. de Vaulezard of France. The contribution which Collins makes to the discussion is the (first?) publication of a demonstration that the construction is valid. Rather than reproduce Collins' graphical argument here, we will provide formulas which mirror the construction and whose justification is somewhat easier to follow. [N.B. For the present form of this article, a graphical construction and its mathematical justification have been added as an addendum.]

As will be seen from these formulas, the ingenious construction works, provided the dialist is able to distinguish, at least roughly, between the East and West sides of the plane with which he or she is working. To eliminate any ambiguity here, we will show that it suffices to record the order in which the three shadow-point observations are made.

I. Necessary Data

Although we do not know the geographic latitude or the reclination or declination of the plane, we do know that it can be treated as a horizontal plane in some latitude (referred to as the plane's latitude), and at some meridian (referred to as the plane's meridian) perhaps different from the geographic meridian. The formulas do not solve for any of the first three of these quantities, but instead determine the latitude and meridian of the plane, so that the assumption may be made that we are dealing with a horizontal dial; the difference in meridians will be found by observing the

meridian of the location (without further reference to the sun) and by constructing the meridian of the plane.

Given the plane's latitude and these two lines (the plane's meridian serving as the substilar line), there are any number of graphical or calculation techniques for finishing the dial. We will concentrate simply on producing these essential elements.

II. Meridian of the Dial's Location

Before presenting the necessary formulas, let us consider how Leybourn determines the meridian of the location. As will be shown later, if we do know the times at which the shadow-points are made, this construction (and section of the present article) is unnecessary and may be replaced by further formulas. Leybourn's technique for determining the meridian of the location is essentially independent of the graphical construction except for its reliance on knowing the position of the dial center; the formulas below will determine the appropriate center point for the dial.

Let a plummet drop from the end-point of the stile until it hits the plane. A line drawn from the center of the dial through the point at which the plummet meets the plane lies in the meridian of the location. If the plane is vertical, so the plummet falls parallel to it, simply drop the plummet from the foot of the stile (assumed to be perpendicular) and trace the line followed by the line of the plummet; this tracing will follow the meridian. Finally, if the plummet does not touch the plane, but the plane is not vertical, then place a straight-edge joining the end-point of the stile to the center of the dial on the plane and drop the plummet from any point on the straight-edge; the point at which the plummet hits the plane is on the meridian line through the center of the dial. As Leybourn points out, in the latter case it may occur that the thread needs to be so close to the plane that it is not feasible to make a good determination of the desired point. Should this problem arise, another possibility is to let the plummet fall from any point on the straight-edge arranged as noted earlier and then to string a thread from the stile's end-point to the plane in such a way that it is perpendicular to the free-hanging plummet line; the point of intersection of the thread with the plane is on the meridian as before described.

III. Modern Reformation of the Solution

Let us now suppose that we are given the points A (the projection perpendicular to the plane of the stile's end-point), B , C and D (the shadow-points recorded at different times in the course of a single day).

Let the length AC be different from the lengths AB and AD ; should this not be possible, then we have the special case of an equatorial dial and the construction is not needed.

Construct an arbitrary rectangular coordinate axis system with the point A as origin and with the perpendicular distance of the stile's end-point above point A as the unit length. The coordinate system will be used to take measurements but neither of the axes needs to be in the meridian.

We first determine values for six angles. Select a ray of one of the axes to serve as the zero-point for measuring angles in the plane. Angles W_b , W_c and W_d are the angles, measured clockwise, that the shadow points make with this ray. The remaining (acute) angles a_b , a_c and a_d are solar altitudes determined by the following:

$$\cot a_b = AB \quad \cot a_c = AC \quad \cot a_d = AD \quad .$$

Evaluate the quantities E , F , G and H as follows:

$$E = \frac{(\cos a_c \cos W_c) - (\cos a_b \cos W_b)}{\sin a_c - \sin a_b}$$

$$F = \frac{(\cos a_c \sin W_c) - (\cos a_b \sin W_b)}{\sin a_c - \sin a_b}$$

$$G = \frac{(\cos a_c \cos W_c) - (\cos a_d \cos W_d)}{\sin a_c - \sin a_d}$$

$$H = \frac{(\cos a_c \sin W_c) - (\cos a_d \sin W_d)}{\sin a_c - \sin a_d}$$

Note that these quantities are always defined because we know a_c is not equal to either a_b or a_d , and therefore the denominators in all cases are nonzero.

We now use the following standard formulas from spherical astronomy:

$$\sin a = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t \quad , \quad \text{and}$$

$$\cos a \cos Z = \sin \varphi \cos \delta \cos t - \cos \varphi \sin \delta$$

where a and Z are the solar altitude and azimuth associated with the shadow-point, φ is the plane's latitude, and δ and t are respectively the solar declination and hour-angle, the latter being determined with respect to the plane's meridian.

Since our shadow points are all made on the same date, we assume no change in solar declination between them; so these formulas tell us that

$$\begin{aligned} \cos a_c \cos Z_c - \cos a_b \cos Z_b &= \sin \varphi \cos \delta (\cos t_c - \cos t_b), & \text{and} \\ \sin a_c - \sin a_b &= \cos \varphi \cos \delta (\cos t_c - \cos t_b). \end{aligned}$$

Hence,

$$\begin{aligned}\tan \varphi &= \frac{\cos a_c \cos Z_c - \cos a_b \cos Z_b}{\sin a_c - \sin a_b} \\ &= \frac{\cos a_c \cos Z_c - \cos a_d \cos Z_d}{\sin a_c - \sin a_d}\end{aligned}$$

We define V as the angle, measured at origin A , between the positive y-axis of the arbitrary system we have adopted and the primary (i.e. North) ray of the plane's meridian. We then have:

$$W_b = V + Z_b \quad , \quad W_c = V + Z_c \quad , \quad W_d = V + Z_d \quad , \quad \text{and}$$

$$\begin{aligned}\cos W &= \cos V \cos Z - \sin V \sin Z \quad , \\ \sin W &= \sin V \cos Z + \cos V \sin Z \quad .\end{aligned}$$

By substituting these equations in the defining equations for E, F, G and H, we have:

$$\begin{aligned}E &= \tan \varphi \cos V - T_b \sin V \\ F &= \tan \varphi \sin V + T_b \cos V \\ G &= \tan \varphi \cos V - T_d \sin V \\ H &= \tan \varphi \sin V + T_d \cos V\end{aligned}$$

where

$$\begin{aligned}T_b &= \frac{\cos a_c \sin Z_c - \cos a_b \sin Z_b}{\sin a_c - \sin a_b} \\ T_d &= \frac{\cos a_c \sin Z_c - \cos a_d \sin Z_d}{\sin a_c - \sin a_d}\end{aligned}$$

We can now achieve our goal of determining the necessary parameters for the sundial as follows:

$$\tan V = (E - G)/(H - F) \quad , \quad \text{and} \quad \tan \varphi = E \cos V + F \sin V \quad .$$

The plane's meridian is the line through A making angle V with the positive y-axis. The dial's center is located on the plane's meridian at a distance from point A equal to $(-\cot \varphi)$; the dial's equinoctial line is perpendicular to the meridian, at a distance from point A equal to $(\tan \varphi)$.

Thus, the locations of the plane's meridian, the dial center, the equinoctial line (and gnomon) are uniquely determined by these equations. Note, however, that these formulas by themselves do not completely determine numeric values for V and φ , since there are two values of V ($-180^\circ < V < 180^\circ$) and hence of φ ($-90^\circ < \varphi < 90^\circ$) which are possible solutions. The two solutions will be for horizontal dials in different hemispheres, the North dial resulting in hour-lines which progress clockwise and the South dial resulting in counter-clockwise hour-lines.

This ambiguity is handled in Leybourn's treatment simply by the following comment:

It is to be observed, that in all Dials the Morning Hours ought to be marked on the West-side, and the Evening Hours on the East.

In actual practice, this caveat may be sufficient, but our calculations do not need to rely on this tacit determination of the cardinal directions if we have noted the order of occurrence of the shadow-points. Given this order and the location of the dial center, we know whether the hour-lines should progress clockwise or not; hence we can select the appropriate pair of values for V and φ and thus correctly determine which ray of the dial's meridian line lies to the North.

We have thus determined the required values for completion of the dial, if we rely on Foster's technique or some similar method for determining the meridian of the location. Note, however, that there is also a calculation option for this placement which we will now develop.

IV. Additional Computations

Consider the following standard formula from spherical astronomy:

$$\cos a \sin Z = \sin t \cos \delta .$$

Combining this equation with the two general equations given in Section III above, yields:

$$\tan t = \frac{\cos a \sin Z}{\cos a \cos Z \sin \varphi + \sin a \cos \varphi}$$

[N.B. This formula produces local apparent time only if the plane's meridian coincides with the meridian of the location. See the discussion to follow.]

We know the solar altitude a and azimuth Z (i.e. $W - V$) associated with each of the three shadow-points, so this equation permits the determination of a time t for each of the readings.

In the seventeenth century, it would not have been appropriate to assume that an accurate independent source for the correct time was available; indeed, the very purpose for performing this construction might be to produce the only such accurate source in a particular location. Today, however, it seems reasonable to assume that an independent source has given a value for the local apparent time of each reading (before the 'corrections' of the equation-of-time or standard or daylight savings adjustments).

The time values generated by the above equation may well differ from the independent values for each reading; however the differences for the three points should all be equal. A difference occurs whenever the meridian of the plane does not coincide with the meridian of the location. The difference in times ($\lambda = \text{Time from above formula} - \text{Independent Local Apparent Time}$), expressed in angular measure, equals the arc angle between the two meridians.

Draw a line through the dial's center point, at an angle N measured clockwise from the primary ray of the plane's meridian, where

$$\tan N = \tan \lambda \sin \phi .$$

This line is the meridian of the location, found without resort to plummet and straight-edge.

The dial is constructed as though the plane is horizontal at latitude ϕ ; markings on its hour-lines are to be offset by a constant amount equal to λ (angular measure). The noon-line of the dial is the meridian of the location as just determined.

Finally, given the information we have developed so far, suppose we wish to determine the reclination and declination of the plane. The following set of equations answers this question if we allow ourselves the luxury of assuming that the geographic latitude ϕ (i.e. the latitude of the location, as opposed to the plane's latitude for which we have already solved) is known.

The plane's reclination R and declination D may be determined as follows:

$$\begin{aligned} \sin R &= \sin \phi \sin \phi + \cos \phi \cos \phi \cos \lambda \\ \tan D &= \frac{\cos \phi \sin \lambda}{\cos \phi \sin \phi \cos \lambda - \sin \phi \sin \phi} \\ \sin D &= (\cos \phi \sin \lambda) / \cos R \end{aligned}$$

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Addendum. Graphical Construction

[This construction and its proof did not appear in the *Bulletin of the British Sundial Society's* version of this paper.]

We now have the task of providing a graphical construction for the plane's meridian and its equinoctial line. What we are given is the point A (the intersection of the plane with a line through the stile's end-point and perpendicular to the plane itself), and the shadow points B , C and D recorded in the course of a single day.

We begin by considering a special case in which points B , C and D lie on a circle with A as center, *i.e.* the lengths of the segments AB , AC and AD are all equal. In this case we have three distinct times in the course of one day at which the sun has the same altitude with respect to the given plane. This situation can occur only if the plane is parallel to the polar horizon; the resulting dial is what is generally referred to as an equatorial dial. The point A is the center of the dial and the hour-points are distributed about A in equal intervals (15° to 1 hour). The meridian of the location determines the noon point, and there is no meridian of the plane.

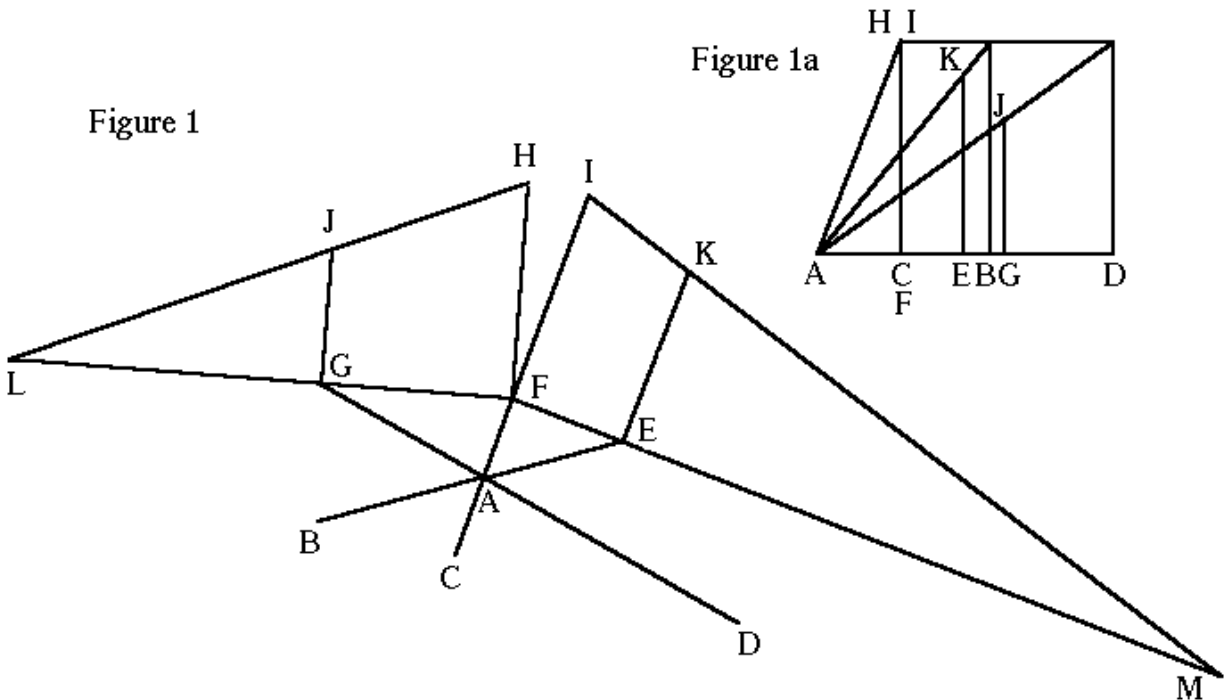
This case is one in which nothing more than a general familiarity with the interrelationships of the points and lines of a common dial is needed; no significant construction has been required.

We now consider the general case in which the points B , C and D do not fall on a circle with center at A . Then at least one of the points, say C , has a distance from A unequal to the corresponding distance of either of the other two points. Of course it may occur that B and D are equidistant from A , in which case we could determine the meridian of the plane as the perpendicular bisector of the angle BAD ; however, we will not rely on this possibility. [See figure 1.]

Construct three right triangles with unit height and with overlapping baselines beginning in a common point. [See figure 1a.] Let the baselines have lengths AC , AB , and AD . Given that C is the point which has the unequal distance from A , we select it as a key point for our construction. Construct a circular arc with center at the triangles' common vertex and radius equal to the hypotenuse of the triangle whose baseline has length AC . From the intersection of this arc with each of the other two hypotenuses, drop perpendiculars to the baselines, thus forming two new right triangles.

The new triangle derived from the original figure with baseline length AB has a baseline length AE and a height EK . Similarly, the other new triangle has baseline and height designated as AG and GJ , respectively.

Extend line CA (figure 1) through A to F such that the length AF equals AC . Similarly extend BA to E so that the new segment has length AE as determined above. Extend DA to G so that the new segment here has length AG .



Draw line FE and two perpendiculars FI and EK to that line such that FI has unit length, and length EK is as determined above.

Draw line FG and two perpendiculars FH and GJ to that line such that FH has unit length, and length GJ is as determined above.

Draw line IK and extend it until it intersects the extended line FE in M . Draw line HJ and extend it until it intersects the extended line FG in L . Connect points L and M to form line LM .

Draw a line FN perpendicular to line LM at N . Draw another line QAO through A and perpendicular to line LM . QAO is the plane's meridian line. The length FN equals the tangent of the plane's latitude. If the length AO is set equal to FN , with O north of A , then a line OT perpendicular to AO at point O is the equinoctial line of the dial.

Let AP be perpendicular to QAO at A and let AP have unit length. Draw line OP and let PQ be perpendicular to OP , intersecting QAO at Q ; then Q is the center of the dial.

Mathematical Demonstration

We will actually provide two demonstrations that the construction is correct. The first follows the analytic approach adopted earlier, considering the coordinates of the various points involved in the procedure. Note, however, that without doubt this approach is not the process which originally gave rise to the construction. There is a more intuitive way to look at the construction, to see how it came about, and to prove that it is correct. This more intuitive approach, making

use of the fact that it comes out of a projective dialing method, will be presented as the second demonstration.

First Demonstration

To demonstrate that the construction is correct, consider the coordinates of the various points as determined by perpendicular axes through point A , with the positive directions of the X and Y axis being to the plane's East and North, respectively. Of course, this assumption may at first appear to beg the question, since we are referring to cardinal points which cannot be determined until the plane's meridian is correctly placed; however, such is not the case. Whether or not we know at this point where the axes are has nothing to do with the fact that they may be assumed to exist and they may be used to keep track of certain relationships between the points. What must be guarded against is any use in the construction of information which is not available independent of any knowledge of the axis placement; since the construction has already been described, it is clear that none of its steps requires any such information. Indeed, it will be incumbent upon the construction to provide this information.

Begin by considering only half of the construction, that half based on the points A , B and C where we know that lengths AB and AC are different. Coordinates of these points are as follows:

	x	y
A	0	0
B	$\cos a_b \sin Z_b / \sin a_b$	$\cos a_b \cos Z_b / \sin a_b$
C	$\cos a_c \sin Z_c / \sin a_c$	$\cos a_c \cos Z_c / \sin a_c$

where a and Z are the altitude and azimuth of the sun with respect to the plane at the time the shadow was cast for the indicated point. From the construction, we further have the points E and F :

	x	y
E	$-\cos a_b \sin Z_b / \sin a_c$	$-\cos a_b \cos Z_b / \sin a_c$
F	$-\cos a_c \sin Z_c / \sin a_c$	$-\cos a_c \cos Z_c / \sin a_c$

Now because of similar triangles in the construction we have:

$$FM / (FM - FE) = FI / EK, \text{ and by construction } FI = 1, EK = \sin a_b / \sin a_c.$$

$$\text{Solving for } FM \text{ yields: } FM = FE \sin a_c / (\sin a_c - \sin a_b).$$

Note that we know we have a non-zero denominator here (*i.e.* the point M of intersection of the lines FM and IM actually exists) because the points C and B were selected so that lengths $AB = \cot a_b$ and $AC = \cot a_c$ would not be equal and therefore a_b and a_c are not equal angles.

Consider now the Y -direction cosine of the directed line segment FE (*i.e.* the cosine of the angle made by FE and a line through F parallel to the Y axis). This cosine is defined as the difference between the Y -coordinate of E and the corresponding coordinate of F , divided by the length FE . Since FM is simply an extension of FE , its analogously defined Y -direction cosine must equal that for FE and we have

$$\frac{(\cos a_c \cos Z_c - \cos a_b \cos Z_b)}{FE \sin a_c} = \frac{\cos a_c \cos Z_c + Y \sin a_c}{FM \sin a_c}$$

where Y is the y -coordinate of the point M . Using this equation and the prior one to solve for Y yields:

$$Y = \frac{\cos a_c \cos Z_c - \cos a_b \cos Z_b}{\sin a_c - \sin a_b} - \frac{\cos a_c \cos Z_c}{\sin a_c}$$

This is the key formula for the construction. Substitute in the first term of the subtraction using the following standard formulas from spherical astronomy:

$$\sin a = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t, \text{ and}$$

$$\cos a \cos Z = \sin \varphi \cos \delta \cos t - \cos \varphi \sin \delta.$$

Having done so, we find that the formula for Y becomes:

$$Y = \tan \varphi - \frac{\cos a_c \cos Z_c}{\sin a_c}.$$

Now take a step back to see what has been accomplished. We find that the value Y of the Y -coordinate of the point M can be expressed independently of the point B . Obviously, the other half of the construction, being carried out with respect to points C and D will have similar results and we will have an equation for the Y -coordinate of point L identical to the already derived equation for point M . In other words, L and M have the same Y -coordinate, and the line LM joining them is therefore perpendicular to the Y -axis. Thus, a line through the origin point A and perpendicular to LM is in fact the Y -axis and is therefore the plane's meridian line. Note that the progression of equations has relied on the construction and has not required evaluation of any of the coordinate points. It is the fact that we have found two points whose Y -coordinates can be shown to be equal (under the coordinate system we are searching for) that allows us to place the Y -axis.

Now that we have the appropriate coordinate axes, let us return to the points F and M . We have formulas for their respective Y -coordinates, and examination will show that the difference between these coordinates equals $\tan \varphi$. Thus, the perpendicular distance from F to the line LM , a distance which equals the difference in Y -coordinates since LM is perpendicular to the Y -axis, is known to be $\tan \varphi$.

Second Demonstration

Consider the points as laid out in figure 1. Suppose the triangles LFH and MFI are rotated 90° so that they are perpendicular to the plane. Note that points H and I come together to represent one point. In this 3-dimensional configuration, consider the distances AH , AK and AJ ; they are all equal (by the construction in figure 1a using a circular arc). Note then that the points H , J and K , with the azimuths and altitudes (EAK , FAH and GAJ from figure 1a) as constructed, represent the locations of the sun on a sphere centered on A at the times when the shadows were cast. Since these shadows all occur on the same day, these three sun locations must be coplanar, and the lines HJL and IKL (in this 3-dimensional configuration) lie in the plane of the sun's daily circle. The points L and M are in both this plane and in the horizontal plane, so they must form an east-west line, parallel to the equinoctial; a line perpendicular to LM through A must be the plane's meridian.

Note also that the sun's daily circle rises above the horizontal plane at an angle equal to $90^\circ - \varphi$, since it is parallel to the equinoctial circle. By drawing FN horizontal and perpendicular to LM at N , we can complete a *vertical* right triangle NFH whose acute angles are $90^\circ - \varphi$ and φ . By construction, the length $FH = 1$, so $FN = \tan \varphi$.

The remainder of the demonstration follows as above.